Extra Lecture: The Structure and Properties of $\mathrm{Irr}(G)$

Goal: Deepen our understanding of $\mathrm{Irr}(G)$, its algebraic structure, orthogonality relations, its role in semisimple decomposition of the group algebra, and its importance in character tables.

1. Definitions and Fundamental Properties

Definition X.1. Let G be a finite group. A *character* $\chi : G \to \mathbb{C}$ is the trace function of a complex representation $\rho : G \to \mathrm{GL}(V)$:

$$\chi(g) = \mathrm{Tr}(\rho(g)).$$

Definition X.2. A character χ is *irreducible* if the corresponding representation ρ is irreducible. The set of all irreducible complex characters of G is denoted:

$$\mathrm{Irr}(G).$$

2. Structure Theorems for $\mathrm{Irr}(G)$

Theorem X.3. Let G be a finite group. Then:

1. $|\mathrm{Irr}(G)| =$ the number of conjugacy classes of G,
2. Every complex representation decomposes uniquely (up to isomorphism) as a direct sum of irreducible representations.

Corollary X.4. The number of distinct irreducible complex representations of G is finite and equals the number of conjugacy classes.

3. Orthogonality Relations

Let $\chi, \psi \in \mathrm{Irr}(G)$. Define the inner product:

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

Theorem X.5 (Orthogonality Relations).

1. $\langle \chi, \psi \rangle = \delta_{\chi\psi}$ (orthonormality),
2. Let $\{C_1, \ldots, C_r\}$ be the conjugacy classes. Then the columns of the character table are orthonormal in the class function inner product.

4. Role in Group Algebra Decomposition

Theorem X.6. Over $\mathbb{C}$, the group algebra decomposes as:

$$\mathbb{C}[G] \cong \bigoplus_{\chi \in \mathrm{Irr}(G)} M_{n_\chi}(\mathbb{C}),$$

where $n_\chi = \chi(1)$ is the degree of χ. Each summand corresponds to the simple component associated with χ.

5. Induced Characters and Completeness

Theorem X.7 (Completeness). The irreducible characters form an orthonormal basis of the space of class functions on G.

Definition X.8. Let $H \leq G$ and $\theta \in \mathrm{Irr}(H)$. The *induced character* $\chi = \mathrm{Ind}_H^G(\theta)$ is given by:

$$\chi(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \theta(x^{-1}gx).$$

Frobenius Reciprocity:

$$\langle \mathrm{Ind}_H^G(\theta), \chi \rangle_G = \langle \theta, \chi|_H \rangle_H.$$

6. Examples

Example X.9 (Irr(S_3)).

Let $G = S_3$. It has 3 conjugacy classes $\Rightarrow |\mathrm{Irr}(S_3)| = 3$. The irreducible characters are:

	[1]	[(12)]	[(123)]
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Note: $1^2 + 1^2 + 2^2 = 6 = |S_3|$.

7. Counterexamples

Counterexample X.10. Two non-isomorphic representations can have the same character over a subfield of $\mathbb{C}$, but not over $\mathbb{C}$ itself. Over $\mathbb{C}$, characters fully determine representations up to isomorphism.

8. Summary

In this extra lecture we reviewed:

- The definition and structure of $\mathrm{Irr}(G)$,
- Orthogonality and inner product structure,
- Their role in group algebra decomposition,
- Use of induction and Frobenius reciprocity.

Next: We'll return to modular theory with *Lecture 7 on Projective Modules.*